# CSIDH:
# An Efficient Post-Quantum Commutative Group Action

https://csidh.isogeny.org


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ECC, Osaka, Japan, 21st November 2018


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# History

1976 Diffie-Hellman: Key exchange using exponentiation in groups (DH)

1985 Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)

1990 Brassard-Yung: Generalizes ‘group exponentiation’ to ‘groups acting on sets’ in a crypto context

1994 Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH

1997 Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)

2003 Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift

# History

2004 Stolbunov-Rostovtsev independently rediscover Couveignes' scheme (CRS)

2006 Charles-Goren-Lauter: Build hash function from supersingular isogeny graph

2010 Childs-Jao-Soukharev: Apply Kuperberg's (and Regev's) hidden shift subexponential quantum algorithm to CRS

2011 Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)

2018 De Feo-Kieffer-Smith: Apply new ideas to speed up CRS

2018 Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)

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- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim$ 85 ms for a full key exchange
- Flexible:
  - Compatible with 0-RTT protocols such as QUIC
  - [DG] uses CSIDH for 'SeaSign' signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange

# CSIDH vs SIDH?

Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison (mostly stolen from Luca de Feo):

| | CSIDH | SIDH |
|---|---|---|
| Speed (NIST 1) | 85ms | $\approx$ 10ms[1] |
| Public key size (NIST 1) | 64B | 378B |
| Key compression (speed) | | $\approx$ 15ms |
| Key compression (size) | | 222B |
| Constant time implementation | yes (quick and dirty) | yes |
| Submitted to NIST | no | yes |
| Maturity | 7 months | 7 years |
| Best classical attack | $p^{1/4}$ | $p^{1/4}$ |
| Best quantum attack | subexponential | $p^{1/6}$ |
| Key size scales | quadratically | linearly |
| Security assumption | isogeny walk problem | ad hoc |
| CPA security | yes | yes |
| CCA security | yes | Fujisaki-Okamoto |
| Non-interactive key exchange | yes | unbearably slow |
| Signatures (classical) | unbearably slow | seconds |
| Signatures (quantum) | seconds | still seconds? |

[1] This is a very conservative estimate!

# Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group $G$ via the map

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x. \end{array}$$

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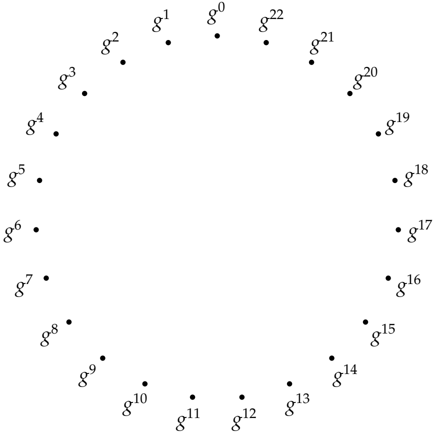
⇝ Idea:

Replace exponentiation on the group $G$ by a group action of a group $H$ on a set $S$:

$$H \times S \to S.$$

# Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes $g^{13}$.



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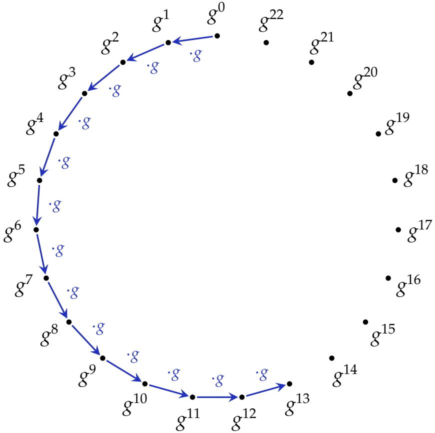

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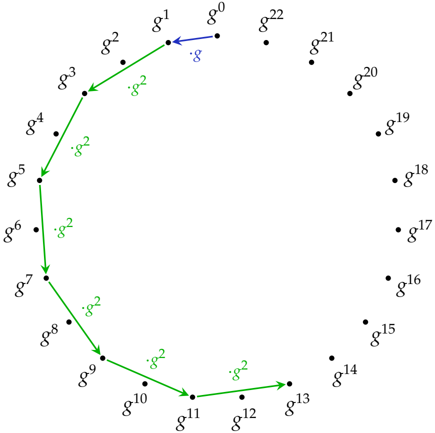

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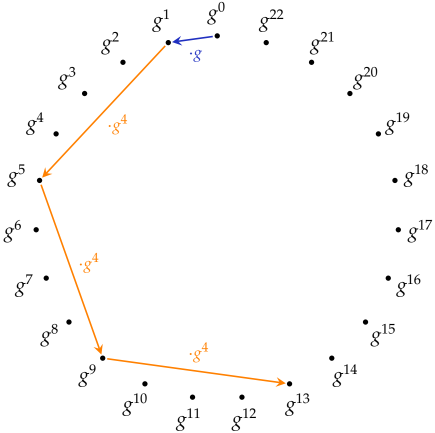

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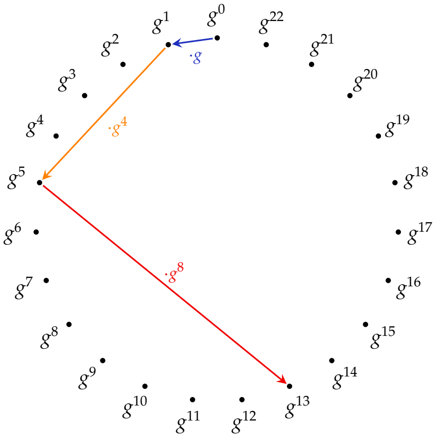

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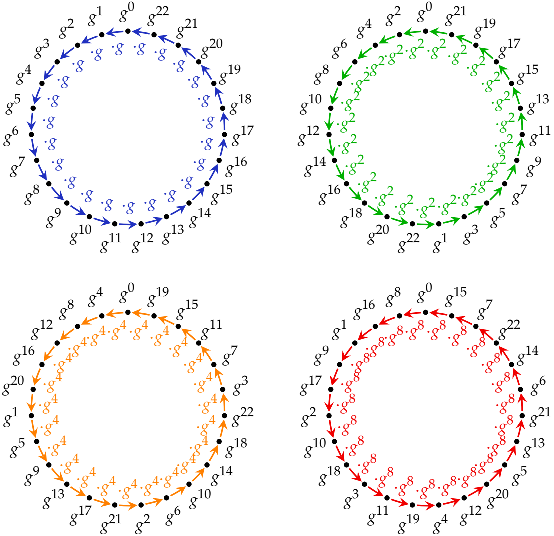

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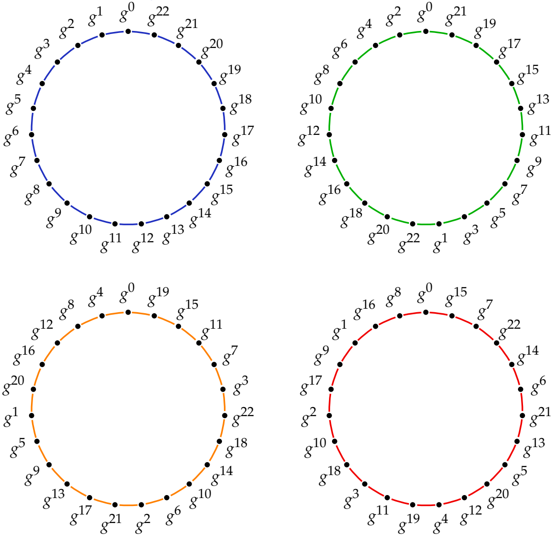

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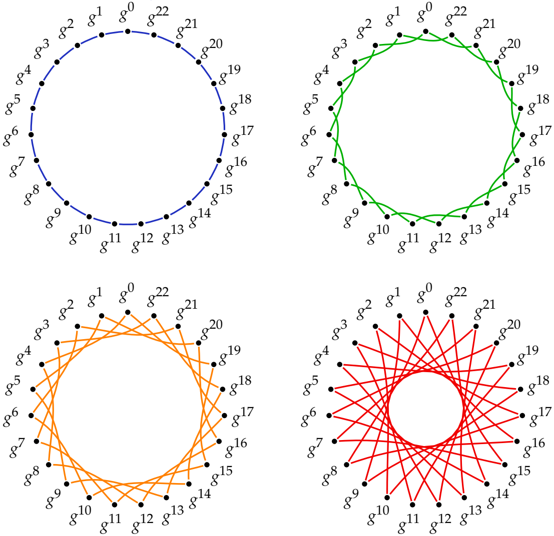

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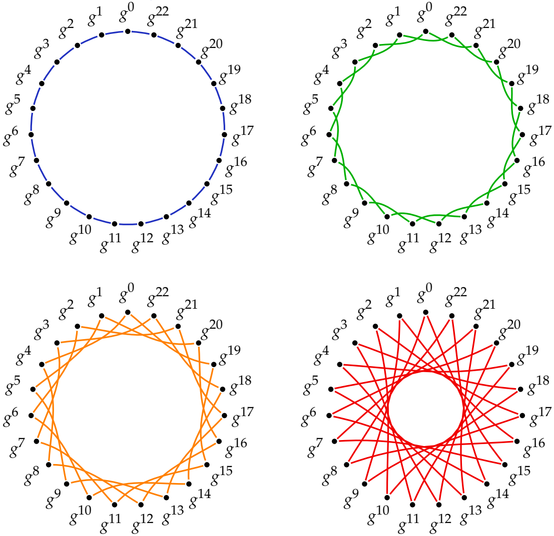
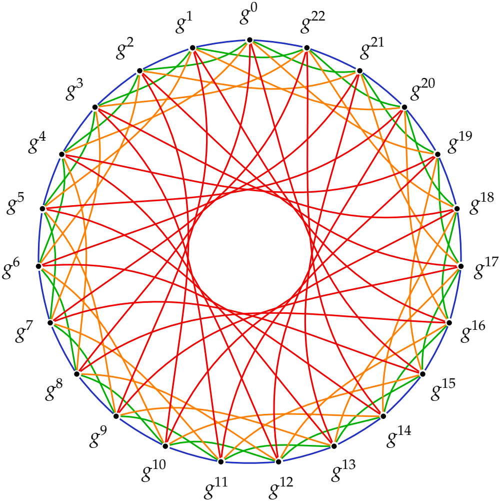



Cycles are compatible: [right, then left] = [left, then right], etc.

# Union of cycles: rapid mixing



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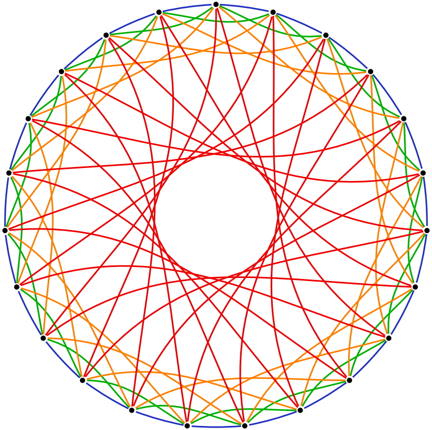
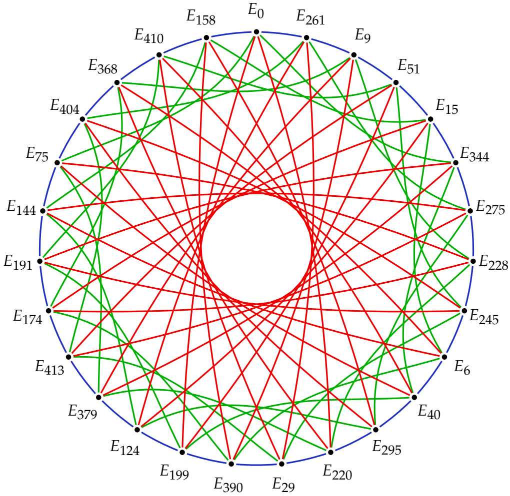


CSIDH: Nodes are now elliptic curves and edges are isogenies.

# Graphs of elliptic curves



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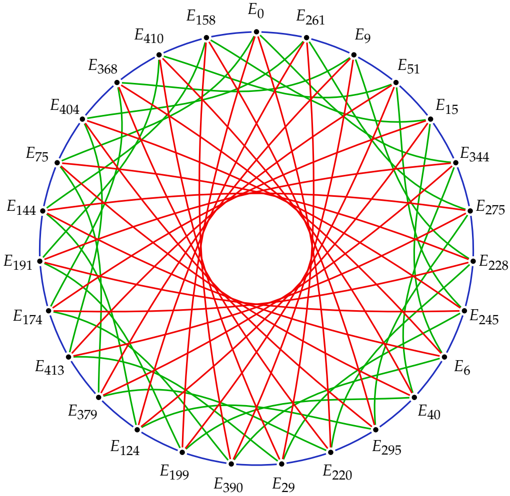


Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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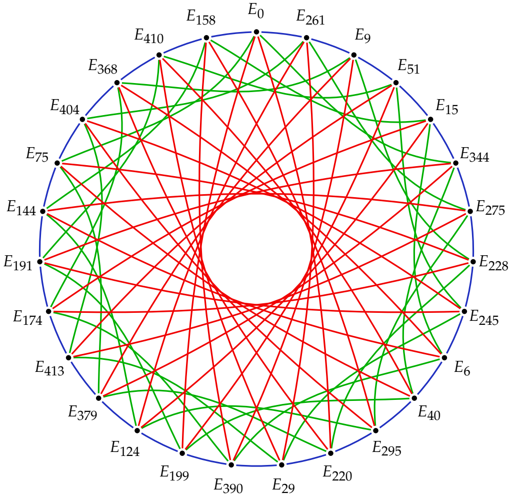



Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies.

# Quantumifying Exponentiation

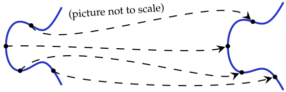
- We want to replace the exponentiation map

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x \end{array}$$

  by a group action on a set.
- Replace $G$ by the set $S$ of supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
- Replace $\mathbb{Z}$ by a commutative group $H$... more details to come!
- The action of a well-chosen $h \in H$ on $S$ moves the elliptic curves one step around one of the cycles.

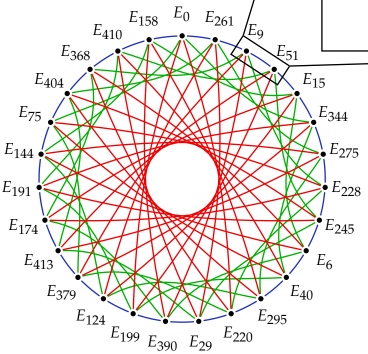
# Graphs of elliptic curves

## A 3-isogeny




$$E_{51}\colon y^2 = x^3 + 51x^2 + x \longrightarrow E_9\colon y^2 = x^3 + 9x^2 + x$$

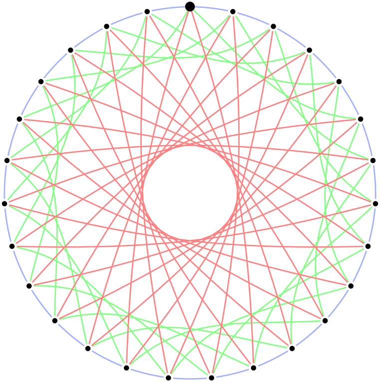
$$(x, y) \longmapsto \left( \frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97},\; y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$



# Diffie-Hellman on 'nice' graphs

Alice

[+, −, +, −]



Bob

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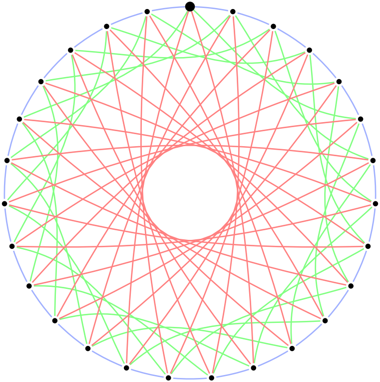

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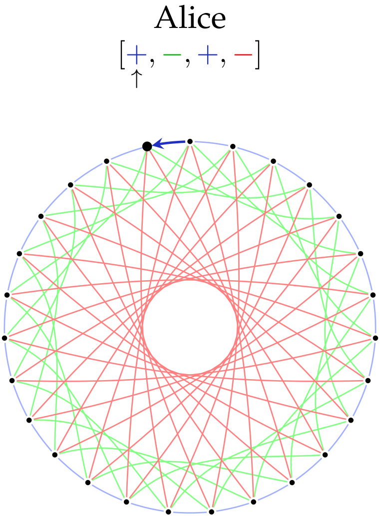


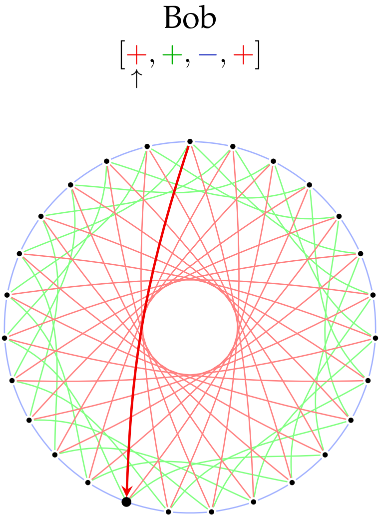

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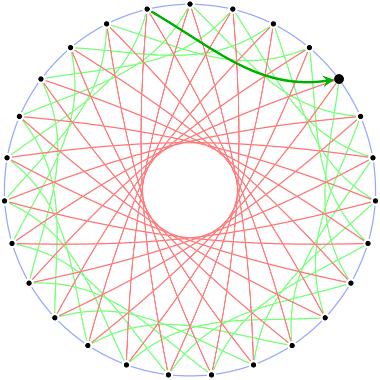

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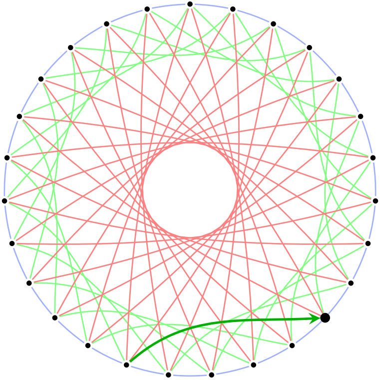

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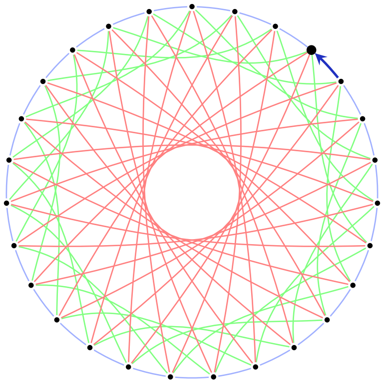

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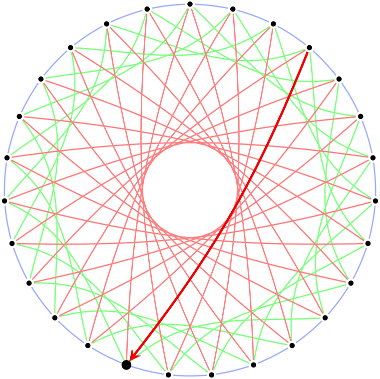

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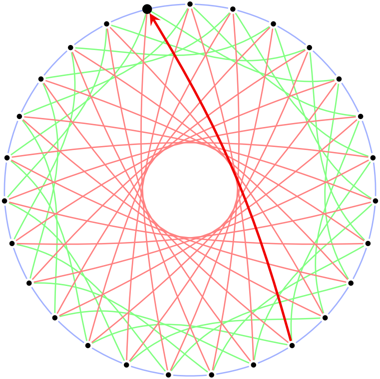

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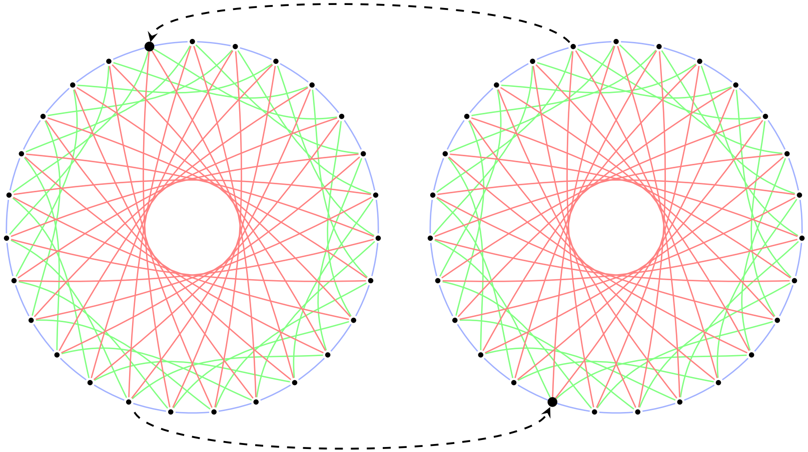

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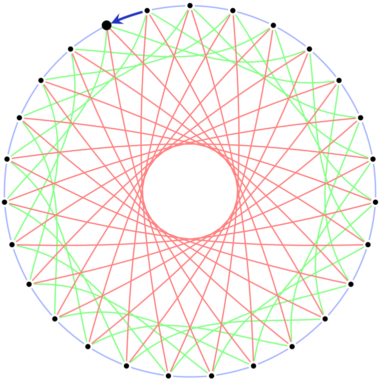

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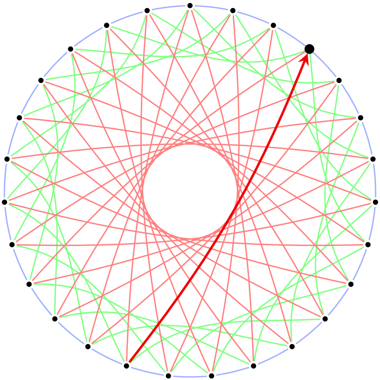

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$\uparrow$

Bob

$[+, +, -, +]$

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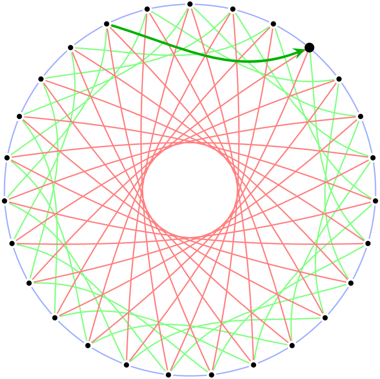

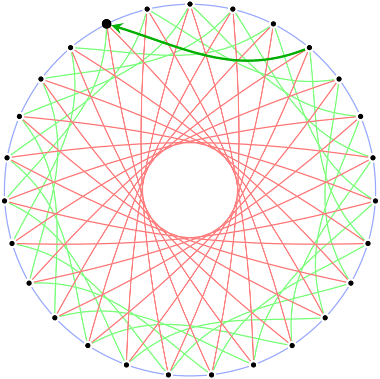

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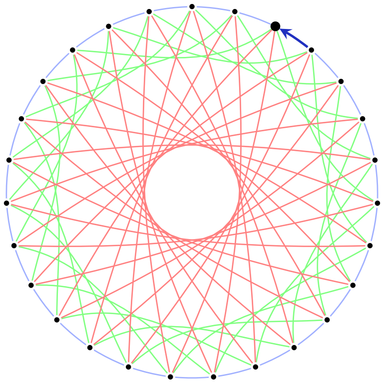

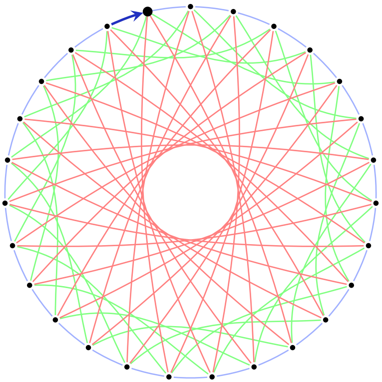

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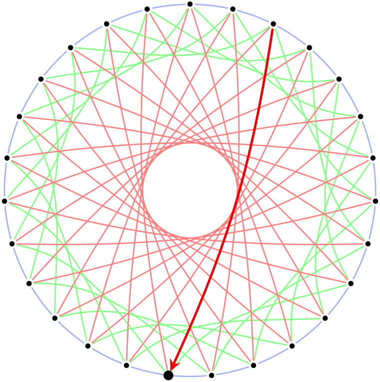

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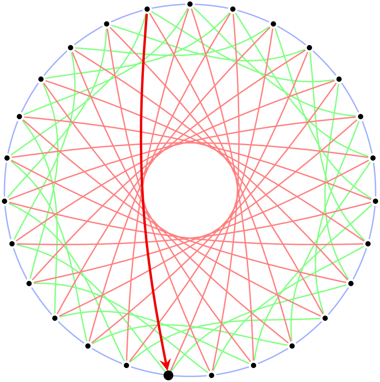

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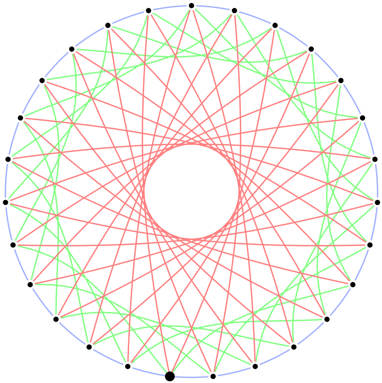

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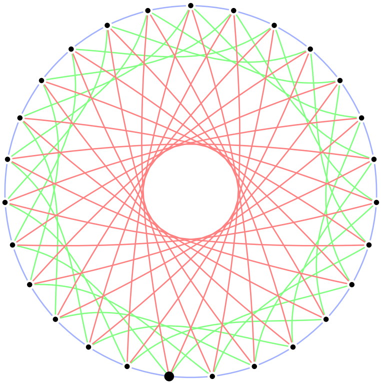

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Important properties for such a walk:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

# Towards IP1: Isogeny graphs

First some reminders (see eg. autumn school slides):

- An elliptic curve $E/\mathbb{F}_p$ (for $p \geq 5$) is supersingular if $\#E(\mathbb{F}_p) = p + 1$.

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- The dual isogeny is also an $\ell$-isogeny.

# Towards IP1: Isogeny graphs

## Definition

Let $p$ and $\ell$ be distinct primes. The isogeny graph $G_\ell$ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$ (up to $\mathbb{F}_p$-isomorphism).
- Edges: we draw an edge $E - E'$ to represent an $\ell$-isogeny $f : E \to E'$ together with its dual $\ell$-isogeny.

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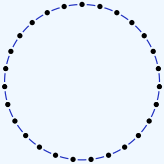
- In our example, these are

$G_3$:



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- In our example, these are

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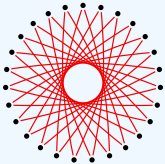

# Towards IP1: Isogeny graphs

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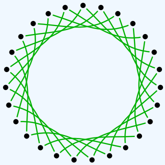
Let $p$ and $\ell$ be distinct primes. The isogeny graph $G_\ell$ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$ (up to $\mathbb{F}_p$-isomorphism).
- Edges: we draw an edge $E - E'$ to represent an $\ell$-isogeny $f : E \to E'$ together with its dual $\ell$-isogeny.

- In our example, these are

$G_7$:



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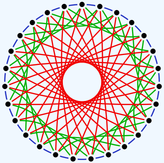

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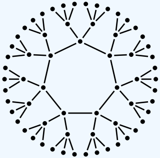
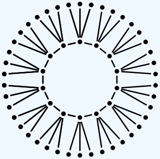
- Generally, the $G_\ell$ look something like

$G_3$:



$G_5$:



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- Two nodes are at different distances from the cycle if and only if they have different endomorphism rings.

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Let $E/\mathbb{F}_p$ be an elliptic curve.

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- The Frobenius map

$$\begin{array}{rccc} \pi: & E & \to & E \\ & (x,y) & \mapsto & (x^p, y^p) \end{array}$$

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### Example

Let $p > 3$, let $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ be a supersingular elliptic curve, and let $\pi$ be the Frobenius endomorphism. Then

$$\pi \circ \pi = [-p]$$

and

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-p}] & \to & \mathrm{End}_{\mathbb{F}_p}(E) \\ n & \mapsto & [n] \\ \sqrt{-p} & \mapsto & \pi \end{array}$$

extends $\mathbb{Z}$-linearly to a ring homomorphism.

# Towards IP1: Group action

For $p \equiv 3 \pmod 8$ and $p \geq 5$, if $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\mathrm{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$.

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- For $[I] \in \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}])$, let $\tilde{I}$ be an integral representative of the ideal class $[I]$. Then
$$\begin{array}{rcl} \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ ([I], E) & \mapsto & f_{H_{\tilde{I}}}(E) \end{array}$$
is a free, transitive group action!

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$\rightsquigarrow$ there is a choice of $\ell_1, \ldots, \ell_n$ such that $G_{\ell_1} \cup \cdots \cup G_{\ell_n}$ is a composition of compatible cycles (IP1).

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- Choosing the direction in the graph corresponds to choosing this sign.

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- For every odd prime $\ell | (p + 1)$, the point $\frac{p+1}{\ell} P$ is a point of order $\ell$.
- Given a $\mathbb{F}_p$-rational point of order $\ell$, the isogeny computations can be done over $\mathbb{F}_p$.

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Given the group action as above, Vélu's formulas give actual isogenies!
With our design choices all isogeny computations are over $\mathbb{F}_p$. [2]

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- Public-key validation: Check that $E_A$ has $p+1$ points.

  Easy Monte-Carlo algorithm: Pick random $P$ on $E_A$ and check $[p+1]P = \infty$.[3]

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- Say Alice's secret is isogeny is of degree $\ell_1^{e_1} \cdots \ell_n^{e_n}$. She knows the path, so can do only small degree isogeny computations, giving complexity $O(\sum e_i \ell_i)$. An attacker has to compute one isogeny of large degree (cf. isogeny evaluation complexity from David Jao's talk).
- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from $E_0$ to $E_A$, whereas an attacker has compute all the possible paths from $E_0$.

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- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from $E_0$ to $E_A$, whereas an attacker has compute all the possible paths from $E_0$.
- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

# Quantum Security

Hidden-shift algorithms: Subexponential complexity (Kuperberg, Regev).

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- Kuperberg later [Kup2] gave more trade-off options for quantum and classical memory vs. time.
- Childs-Jao-Soukharev [CJS] applied Kuperberg/Regev to CRS – their attack also applies to CSIDH.
- Part of CJS attack computes many paths in superposition.

# Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

  (and much more).

[4] From [BLMP], using query count of [BS]. [BS] also study quantum evaluation of isogenies but their current preprint misses some costs.

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- For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about $2^{81}$ qubit operations.[4]

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# Parameters

| CSIDH-log $p$ | intended NIST level | public key size | private key size | time (full exchange) | cycles (full exchange) | stack memory | classical security |
|---|---|---|---|---|---|---|---|
| CSIDH-512 | 1 | 64 b | 32 b | 85 ms | 212e6 | 4368 b | 128 |
| CSIDH-1024 | 3 | 128 b | 64 b | | | | 256 |
| CSIDH-1792 | 5 | 224 b | 112 b | | | | 448 |

# Work in progress & future work

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- More applications.
- [Your paper here!]


Thank you!

# References

Mentioned in this talk:


BLMP Bernstein, Lange, Martindale, and Panny:
*Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies*
`https://quantum.isogeny.org`

BS Bonnetain, Schrottenloher:
*Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes*
`https://ia.cr/2018/537`

CLMPR Castryck, Lange, Martindale, Panny, Renes:
*CSIDH: An Efficient Post-Quantum Commutative Group Action*
`https://ia.cr/2018/383`

CJS Childs, Jao, and Soukharev:
*Constructing elliptic curve isogenies in quantum subexponential time*
`https://arxiv.org/abs/1012.4019`

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Mentioned in this talk (contd.):


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`https://ia.cr/2018/648`

FTY Fujioka, Takashima, and Yoneyama:
*One-Round Authenticated Group Key Exchange from Isogenies*
`https://eprint.iacr.org/2018/1033`

MR Meyer, Reith:
*A faster way to the CSIDH*
`https://ia.cr/2018/782`

Kup1 Kuperberg:
*A subexponential-time quantum algorithm for the dihedral hidden subgroup problem*
`https://arxiv.org/abs/quant-ph/0302112`

Kup2 Kuperberg:
*Another subexponential-time quantum algorithm for the dihedral hidden subgroup problem*
`https://arxiv.org/abs/1112.3333`

Reg Regev:
*A subexponential time algorithm for the dihedral hidden subgroup problem with polynomial space*
`https://arxiv.org/abs/quant-ph/0406151`

# References

Further reading:


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*A note on the security of CSIDH*
`https://arxiv.org/pdf/1806.03656`

DPV Decru, Panny, and Vercauteren:
*Faster SeaSign signatures through improved rejection sampling*
`https://eprint.iacr.org/2018/1109`

JLLR Jao, LeGrow, Leonardi, Ruiz-Lopez:
*A polynomial quantum space attack on CRS and CSIDH*
(MathCrypt 2018)



Credits: thanks to Lorenz Panny for many of these slides, including all of the beautiful pictures.